Solution to Homework Assignment for Circuit QED

If you have any questions or find typos, please don't hesitate to contact Leo DiCarlo (l.dicarlo@tudelft.nl).

Problem 1: *Dispersive circuit QED hamiltonian.*
Consider a qubit coupled to a single-mode resonator. As Andreas described, the system is governed by the Jaynes-Cummings hamiltonian

$$H_{\mathrm{JC}} = \hbar\omega_{\mathrm{r}}\left(n + \frac{1}{2}\right) - \hbar\frac{\omega_{\mathrm{a}}}{2}\sigma_z + \hbar g\left(a\sigma_+ + a^\dagger\sigma_-\right),$$

where $a^\dagger$ and a are creation and annihilation operators for the mode, $n = a^\dagger a$, and σ_z, $\sigma_+ = \frac{1}{2}(\sigma_x - i\sigma_y)$ and $\sigma_- = \frac{1}{2}(\sigma_x + i\sigma_y)$ are qubit operators. To ensure we use the same conventions, let us specify the Pauli operators:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In this problem, we derive an approximate hamiltonian valid in the dispersive regime, where the qubit and resonator are off-resonant and their detuning $\Delta = \omega_{\mathrm{a}} - \omega_{\mathrm{r}}$ is large (in an absolute sense) relative to the dipolar coupling strength, i.e., $g/|\Delta| \ll 1$. Consider the unitary transformation:

$$U = \exp\frac{g}{\Delta}\left(a\sigma_+ - a^\dagger\sigma_-\right) = I + \frac{g}{\Delta}\left(a\sigma_+ - a^\dagger\sigma_-\right) + \frac{1}{2}\frac{g^2}{\Delta^2}\left(a\sigma_+ - a^\dagger\sigma_-\right)^2 + \cdots$$

- As a warmup, show that the second-order term of U can be rewritten as

$$-\frac{1}{2}\frac{g^2}{\Delta^2}\left(n - \frac{1}{2}\sigma_z + \frac{1}{2}I\right).$$

Hints: make use of the commutation relation $\left[a, a^\dagger\right] = 1$ and $\sigma_-^2 = \sigma_+^2 = 0$.

Start by expanding the second-order term:

$$\left(a\sigma_+ - a^\dagger\sigma_-\right)^2 = a^2\sigma_+^2 - a^{\dagger 2}\sigma_-^2 - aa^\dagger\sigma_+\sigma_- - a^\dagger a\sigma_+\sigma_-,$$

where we have taken into account that qubit and resonator operators commute. The first two terms in the RHS drop out as $\sigma_-^2 = \sigma_+^2 = 0$. Next, making use of the commutation relation $\left[a, a^\dagger\right] = aa^\dagger - a^\dagger a = 1$,

$$\begin{aligned}\left(a\sigma_+ - a^\dagger\sigma_-\right)^2 &= -(1+n)\,\sigma_+\sigma_- - n\sigma_+\sigma_- \\ &= -n\left(\sigma_+\sigma_- + \sigma_+\sigma_-\right) - \sigma_+\sigma_-. \end{aligned} \tag{1}$$

Now we simplify the qubit terms:

$$\begin{aligned}\sigma_+\sigma_- + \sigma_+\sigma_- &= \frac{1}{4}\left(\sigma_x - i\sigma_y\right)\left(\sigma_x - i\sigma_y\right) + \frac{1}{4}\left(\sigma_x - i\sigma_y\right)\left(\sigma_x + i\sigma_y\right) \\ &= \frac{1}{2}\left(\sigma_x^2 + \sigma_y^2\right) \\ &= \frac{1}{2}\left(I + I\right) \\ &= I \end{aligned} \tag{2}$$

and

$$\begin{aligned}\sigma_+\sigma_- &= \frac{1}{4}\left(\sigma_x - i\sigma_y\right)\left(\sigma_x + i\sigma_y\right) \\ &= \frac{1}{4}\left(\sigma_x^2 + \sigma_y^2 + i\sigma_x\sigma_y - i\sigma_y\sigma_x\right) \\ &= \frac{1}{4}\left(2I + i\left(i\sigma_z\right) - i\left(-i\sigma_z\right)\right) \\ &= \frac{1}{2}\left(I - \sigma_z\right). \end{aligned} \tag{3}$$

Combining Eqns. (1) to (3), we obtain the sought-after expression for the second-order term of U.

Consider the transformed hamiltonian given by

$$H = UH_{\mathrm{JC}}U^\dagger.$$

We will approximate H by keeping only terms to first order in the small parameter g/Δ. Let us do this methodically by transforming each term in H_{JC}. The transformation of the vacuum energy is trivial:

$$U\left(\frac{1}{2}\hbar\omega_{\mathrm{r}}\right)U^\dagger = \frac{1}{2}\hbar\omega_{\mathrm{r}}$$

- Now on to the other resonator term:

$$U\left(\hbar\omega_{\mathrm{r}}n\right)U^\dagger \approx \hbar\omega_{\mathrm{r}}n + \hbar\omega_{\mathrm{r}}\frac{g}{\Delta}\,\underline{\hspace{3cm}} + \hbar\omega_{\mathrm{r}}\frac{g^2}{\Delta^2}\,\underline{\hspace{3cm}}$$

We start with the first-order term. Taking into account the prefactor,

$$\underline{\hspace{3cm}} = \left(a\sigma_+ - a^\dagger\sigma_-\right)n - n\left(a\sigma_+ - a^\dagger\sigma_-\right) = \left[a, n\right]\sigma_+ - \left[a^\dagger, n\right]\sigma_-. \tag{4}$$

We can simplify the two commutators:

$$\left[a, n\right] = aa^\dagger a - a^\dagger aa = \left[a, a^\dagger\right]a = a \tag{5}$$

and

$$\left[a^\dagger, n\right] = a^\dagger a^\dagger a - a^\dagger a a^\dagger d = a^\dagger \left[a^\dagger, a\right] = -a^\dagger \left[a, a^\dagger\right] = -a^\dagger. \tag{6}$$

Combining Eqns. (4) to (6) we get

$$U\left(\hbar\omega_{\mathrm{r}} n\right) U^\dagger \approx \hbar\omega_{\mathrm{r}} n + \hbar\omega_{\mathrm{r}} \frac{g}{\Delta}\left(a\sigma_+ + a^\dagger\sigma_-\right) + \hbar\omega_{\mathrm{r}} \frac{g^2}{\Delta^2} \underline{\hspace{3cm}}.$$

Let's now consider the three second-order terms. Taking into account the prefactor, we have

$$\underline{\hspace{3cm}} = -\frac{1}{4}\left(2n - \sigma_z + I\right) n - n\frac{1}{4}\left(2n - \sigma_z + I\right) - \left(a\sigma_+ - a^\dagger\sigma_-\right) n \left(a\sigma_+ - a^\dagger\sigma_-\right).$$

Simplify the first two sets of terms:

$$\underline{\hspace{3cm}} = -n^2 + \frac{1}{2} n\sigma_z - \frac{1}{2} n - \left(a\sigma_+ - a^\dagger\sigma_-\right) n \left(a\sigma_+ - a^\dagger\sigma_-\right). \tag{7}$$

Let's now focus on the third set of terms:

$$\begin{aligned} -\left(a\sigma_+ - a^\dagger\sigma_-\right) n \left(a\sigma_+ - a^\dagger\sigma_-\right) &= a n a^\dagger \sigma_+\sigma_- + a^\dagger n a \sigma_-\sigma_+ \\ &= \left(n^2 + 2n + 1\right)\sigma_+\sigma_- + \left(n^2 - n\right)\sigma_-\sigma_+ \\ &= \left(n^2 + 2n + 1\right)\frac{1}{2}\left(I - \sigma_z\right) + \left(n^2 - n\right)\frac{1}{2}\left(I + \sigma_z\right) \\ &= n^2 + \frac{1}{2} n + \frac{1}{2} - \frac{3}{2} n\sigma_z - \frac{1}{2}\sigma_z. \end{aligned} \tag{8}$$

Here, we have used the relations:

$$a n a^\dagger = a a^\dagger a a^\dagger = \left(1 + a^\dagger a\right)\left(1 + a^\dagger a\right) = (1+n)^2 = n^2 + 2n + 1$$

and

$$a^\dagger n a = a^\dagger a^\dagger a a = a^\dagger\left(a a^\dagger - 1\right) a = n^2 - n.$$

Combining Eqns. (7) and (8), we arrive at

$$U\left(\hbar\omega_{\mathrm{r}} n\right) U^\dagger \approx \hbar\omega_{\mathrm{r}} n + \hbar\omega_{\mathrm{r}} \frac{g}{\Delta}\left(a\sigma_+ + a^\dagger\sigma_-\right) + \hbar\omega_{\mathrm{r}} \frac{g^2}{\Delta^2}\left(-n\sigma_z - \frac{1}{2}\sigma_z + \frac{1}{2}\right). \tag{9}$$

- Now on to the qubit term:

$$U\left(-\frac{1}{2}\hbar\omega_{\mathrm{a}}\sigma_z\right) U^\dagger \approx -\frac{1}{2}\hbar\omega_{\mathrm{a}}\sigma_z + \frac{1}{2}\hbar\omega_{\mathrm{a}}\frac{g}{\Delta}\underline{\hspace{3cm}} + \frac{1}{2}\hbar\omega_{\mathrm{a}}\frac{g^2}{\Delta^2}\underline{\hspace{3cm}}$$

We start with the two first-order terms. Taking into account the prefactor,

$$\begin{aligned} \underline{\hspace{3cm}} &= -\left(a\sigma_+ - a^\dagger\sigma_-\right)\sigma_z + \sigma_z\left(a\sigma_+ - a^\dagger\sigma_-\right) \\ &= a\left[\sigma_z, \sigma_+\right] + a^\dagger\left[\sigma_-, \sigma_z\right] = -2a\sigma_+ - 2a^\dagger\sigma_-. \end{aligned}$$

Thus,

$$U\left(-\frac{1}{2}\hbar\omega_{\mathrm{a}}\sigma_z\right)U^\dagger \approx -\frac{1}{2}\hbar\omega_{\mathrm{a}}\sigma_z - \hbar\omega_{\mathrm{a}}\frac{g}{\Delta}\left(a\sigma_+ + a^\dagger\sigma_-\right) + \frac{1}{2}\hbar\omega_{\mathrm{a}}\frac{g^2}{\Delta^2}\underline{\hspace{3cm}}.$$

Let's now consider the three second-order terms. Taking into account the prefactor, we have

$$\begin{aligned} \underline{\hspace{3cm}} &= \frac{1}{4}\left(2n - \sigma_z + I\right)\sigma_z + \sigma_z\frac{1}{4}\left(2n - \sigma_z + I\right) + \left(a\sigma_+ - a^\dagger\sigma_-\right)\sigma_z\left(a\sigma_+ - a^\dagger\sigma_-\right) \\ &= n\sigma_z + \frac{1}{2}\sigma_z - \frac{1}{2} + \left(a\sigma_+ - a^\dagger\sigma_-\right)\sigma_z\left(a\sigma_+ - a^\dagger\sigma_-\right), \end{aligned} \tag{10}$$

where we've made use of $\sigma_z^2 = I$. Let's focus on the third set of terms,

$$\begin{aligned} \left(a\sigma_+ - a^\dagger\sigma_-\right)\sigma_z\left(a\sigma_+ - a^\dagger\sigma_-\right) &= -\sigma_z\left(a\sigma_+ - a^\dagger\sigma_-\right)\left(a\sigma_+ - a^\dagger\sigma_-\right) \\ &= aa^\dagger\sigma_z\sigma_+\sigma_- + a^\dagger a\sigma_z\sigma_-\sigma_+ \\ &= \left(1+n\right)\sigma_z\frac{1}{2}\left(I - \sigma_z\right) + n\sigma_z\frac{1}{2}\left(I + \sigma_z\right) \\ &= \left(1+n\right)\frac{1}{2}\left(\sigma_z - I\right) + n\frac{1}{2}\left(I + \sigma_z\right) \\ &= n\sigma_z + \frac{1}{2}\sigma_z - \frac{1}{2}. \end{aligned} \tag{11}$$

Combining Eqns. (10) and (11),

$$U\left(-\frac{1}{2}\hbar\omega_{\mathrm{a}}\sigma_z\right)U^\dagger \approx -\frac{1}{2}\hbar\omega_{\mathrm{a}}\sigma_z - \hbar\omega_{\mathrm{a}}\frac{g}{\Delta}\left(a\sigma_+ + a^\dagger\sigma_-\right) + \hbar\omega_{\mathrm{a}}\frac{g^2}{\Delta^2}\left(n\sigma_z + \frac{1}{2}\sigma_z - \frac{1}{2}\right). \tag{12}$$

- And finally to the interaction term:

$$U\hbar g\left(a\sigma_+ + a^\dagger\sigma_-\right)U^\dagger \approx \hbar g\left(a\sigma_+ + a^\dagger\sigma_-\right) + \hbar g\frac{g}{\Delta}\underline{\hspace{3cm}}.$$

$$\begin{aligned} \underline{\hspace{3cm}} &= \left(a\sigma_+ - a^\dagger\sigma_-\right)\left(a\sigma_+ + a^\dagger\sigma_-\right) - \left(a\sigma_+ + a^\dagger\sigma_-\right)\left(a\sigma_+ - a^\dagger\sigma_-\right) \\ &= aa^\dagger\sigma_+\sigma_- - a^\dagger a\sigma_-\sigma_+ + aa^\dagger\sigma_+\sigma_- - a^\dagger a\sigma_-\sigma_+ \\ &= 2\left(1+n\right)\frac{1}{2}\left(I - \sigma_z\right) - 2n\frac{1}{2}\left(I + \sigma_z\right) \\ &= -2n\sigma_z - \sigma_z + 1. \end{aligned}$$

Thus,

$$U\hbar g\left(a\sigma_+ + a^\dagger\sigma_-\right)U^\dagger \approx \hbar g\left(a\sigma_+ + a^\dagger\sigma_-\right) - 2\hbar\frac{g^2}{\Delta}\left(n\sigma_z + \frac{1}{2}\sigma_z - \frac{1}{2}\right). \tag{13}$$

- Show that the combination of all these terms simplifies to the hamiltonian given in Lecture 1:

$$H \approx \hbar\omega_{\rm r}\left(n + \frac{1}{2}\right) - \frac{1}{2}\hbar\left(\omega_{\rm a} + \frac{g^2}{\Delta}\right)\sigma_z - \hbar\frac{g^2}{\Delta}n\sigma_z + \frac{1}{2}\hbar\frac{g^2}{\Delta}.$$

  Combining Eqns. (9), (12) and (13) with the relation $\Delta = \omega_{\rm a} - \omega_{\rm r}$ gives precisely this result!

- Identify the Lamb and AC-Stark shifts.

  The term $-\hbar\frac{g^2}{2\Delta}\sigma_z$ is the Lamb shift and the term $-\hbar\frac{g^2}{\Delta}n\sigma_z$ is the AC-Stark shift.

- Which term do we exploit to perform qubit readout in dispersive circuit QED?

  For readout, we exploit the qubit-state dependence of the resonator frequency. Thus, we exploit the AC-Stark shift term.

Problem 2: *Resonator-mediated qubit-qubit interaction.*
Let us consider now a system of two qubits (labelled 1 and 2) that both couple to a common single-mode resonator. The qubits have no direct interaction between them. The Jaynes-Cummings hamiltonian is expanded into a Tavis-Cummings hamiltonian:

$$H_{\rm TC} = \hbar\omega_{\rm r}\left(n + \frac{1}{2}\right) - \sum_{i\in\{1,2\}}\hbar\frac{\omega_{{\rm a},i}}{2}\sigma_{z,i} + \hbar g_i\left(a\sigma_{+,i} + a^\dagger\sigma_{-,i}\right),$$

We assume both qubits are dispersive with respect to the resonator, i.e., $g_{1(2)}/|\Delta_{1(2)}| \ll 1$, where $\Delta_i = \omega_{{\rm a},i} - \omega_{\rm r}$. As in the previous problem, we can derive an approximate hamiltonian for this regime, now using the transformation

$$U = \exp\left(\frac{g_1}{\Delta_1}\left(a\sigma_{+,1} - a^\dagger\sigma_{-,1}\right) + \frac{g_2}{\Delta_2}\left(a\sigma_{+,2} - a^\dagger\sigma_{-,2}\right)\right).$$

For this problem, let us focus on the transformation of the interaction term only:

$$U\left(\hbar g_1\left(a\sigma_{+,1} + a^\dagger\sigma_{-,1}\right) + \hbar g_2\left(a\sigma_{+,2} + a^\dagger\sigma_{-,2}\right)\right)U^\dagger$$

- Show that in addition to the terms for each qubit (similar to those in Problem 1), there appears a qubit-qubit interaction of the form

$$\hbar J\left(\sigma_{-,1}\sigma_{+,2}+\sigma_{+,1}\sigma_{-,2}\right).$$

Jargon: we say that the cavity *mediates* a *swap* (also called *exchange*) interaction between the two qubits.

The terms we are interested in have the form

$$\left(\frac{g_i}{\Delta_i}\left(a\sigma_{+,i}-a^\dagger\sigma_{-,i}\right)\right)\left(\hbar g_j\left(a\sigma_{+,j}+a^\dagger\sigma_{-,j}\right)\right)+\left(\hbar g_j\left(a\sigma_{+,j}+a^\dagger\sigma_{-,j}\right)\right)\left(-\frac{g_i}{\Delta_i}\left(a\sigma_{+,i}-a^\dagger\sigma_{-,i}\right)\right).$$

Terms with $i=j$ lead to the terms for each qubit that we are familiar with from Problem 1. So, let's consider the remaining cases $i\neq j$. Rearranging terms:

$$\hbar\frac{g_ig_j}{\Delta_i}\left(\left(a\sigma_{+,i}-a^\dagger\sigma_{-,i}\right)\left(a\sigma_{+,j}+a^\dagger\sigma_{-,j}\right)-\left(a\sigma_{+,j}+a^\dagger\sigma_{-,j}\right)\left(a\sigma_{+,i}-a^\dagger\sigma_{-,i}\right)\right).$$

Several terms directly cancel each other, leaving us with

$$\hbar\frac{g_ig_j}{\Delta_i}\left(aa^\dagger\sigma_{+,i}\sigma_{-,j}-a^\dagger a\sigma_{-,i}\sigma_{+,j}+aa^\dagger\sigma_{+,j}\sigma_{-,i}-a^\dagger a\sigma_{-,j}\sigma_{+,i}\right).$$

Operators acting on different qubits commute, so this simplifies to

$$\hbar\frac{g_ig_j}{\Delta_i}\left(\left(aa^\dagger-a^\dagger a\right)\sigma_{+,i}\sigma_{-,j}+\left(aa^\dagger-a^\dagger a\right)\sigma_{-,i}\sigma_{+,j}\right).$$

Making use of the commutation relation $[a,a^\dagger]=1$, we arrive at

$$\hbar\frac{g_ig_j}{\Delta_i}\left(\sigma_{+,i}\sigma_{-,j}+\sigma_{-,i}\sigma_{+,j}\right).$$

Considering the two cases $i=1,j=2$ and $i=2,j=1$, we reach our final result

$$\hbar g_1g_2\left(\frac{1}{\Delta_1}+\frac{1}{\Delta_2}\right)\left(\sigma_{+,1}\sigma_{-,2}+\sigma_{-,1}\sigma_{+,2}\right). \tag{14}$$

- Give an expression for J in terms of g_1, g_2, Δ_1, and Δ_2.
Note: this expression for J will differ from that given in the lecture slides. The reason is that the transformations of the resonator and qubit terms combine to give more terms of this form.

From Eqn. (14), we read off

$$J=g_1g_2\left(\frac{1}{\Delta_1}+\frac{1}{\Delta_2}\right).$$

Note indeed that this expression differs by a factor of 2 from that in the lecture slides!