Preparation

```
SetOptions[Plot, Frame → True,
  FrameStyle → Directive[Thick, Black, 15], ImageSize → 290];
SetOptions[ListPlot, Frame → True,
  FrameStyle → Directive[Thick, Black, 15], ImageSize → 290];
```

Exercise 1

(a) Write down the Hamiltonian for the two - dimensional qubit subspace in terms of the Pauli matrices σx and σz by restricting the quantum states to n = 0, 1.

(i) The Hamiltonian of a Cooper - pair box for the lowest two charge states can be written as

$$H_{CPB} = \begin{pmatrix} Ec\,(0-ng)^2 & -\frac{Ej}{2} \\ -\frac{Ej}{2} & Ec\,(1-ng)^2 \end{pmatrix} = \begin{pmatrix} En_1 & -\frac{Ej}{2} \\ -\frac{Ej}{2} & En_2 \end{pmatrix}$$

with $En_i = Ec\,((i-1)-ng)^2$ the (kinetic) energy of the bare states.

(ii) we subtract the reference level lying between the two states $E_{ref} = -\frac{En_1 + En_2}{2} = -\frac{\Delta}{2}$

$$H_{CPB} = \frac{1}{2}\begin{pmatrix} Ec\left((0-ng)^2-(1-ng)^2\right) & -Ej \\ -Ej & Ec\left((1-ng)^2-(0-ng)^2\right) \end{pmatrix} =$$

$$\frac{1}{2}\begin{pmatrix} -\Delta & -Ej \\ -Ej & \Delta \end{pmatrix} = -\frac{\Delta}{2}\sigma_z - \frac{E_j}{2}\sigma_z$$

(b) What is the transition frequency between ground and excited state as a function of ng?

(i) we diagonalize H_{CPB} with reference level being 0 with simple matrix diagonalization :

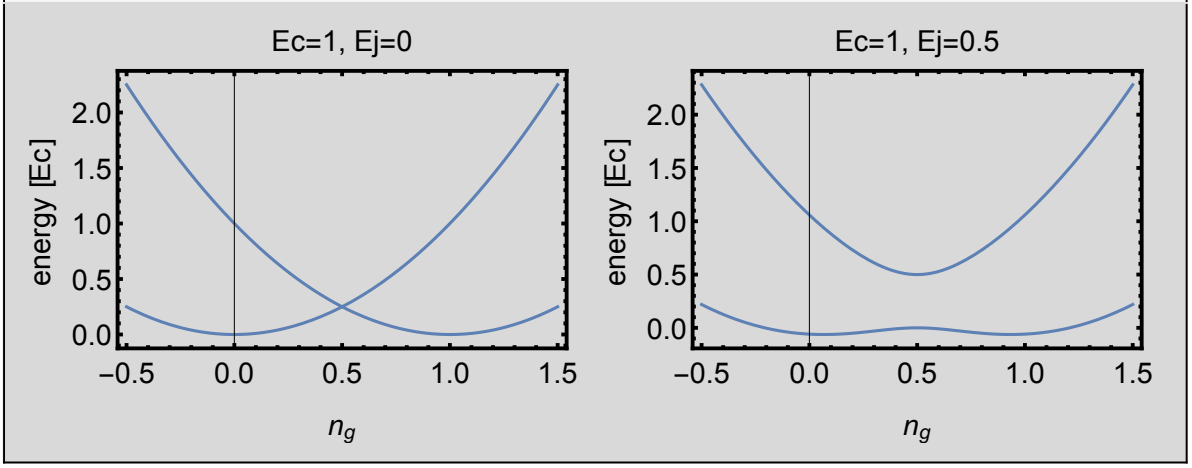
The energy levels of the ground and excited states are

$$\frac{1}{2}\left(-\sqrt{Ej^2 + Ec^2\,(1-2\,ng)^2} + Ec\,(1+2\,(-1+ng)\,ng)\right)$$

and

$$\frac{1}{2}\left(\sqrt{Ej^2 + Ec^2\,(1-2\,ng)^2} + Ec\,(1+2\,(-1+ng)\,ng)\right)$$ respectively.

```
Grid[{{
   Plot[{EigenSystem[[1]][[1]], EigenSystem[[1]][[2]]} /. {Ec → 1, Ej → 0}, {ng,
     -0.5, 1.5}, FrameLabel → {{"energy [Ec]", None}, {"n_g", "Ec=1, Ej=0"}}],
   Plot[{EigenSystem[[1]][[1]], EigenSystem[[1]][[2]]} /. {Ec → 1, Ej → 0.5},
    {ng, -0.5, 1.5},
    FrameLabel → {{"energy [Ec]", None}, {"n_g", "Ec=1, Ej=0.5"}}]
  }}]
```



(ii) we diagonalize H_{CPB} with reference level between the eigenstates with simple matrix diagonalization :

```
EigenSystem = MatrixForm[FullSimplify[
     Eigensystem[ 1/2 ( Ec((0 - ng)^2 - (1 - ng)^2)      -Ej
                        -Ej                              Ec((1 - ng)^2 - (0 - ng)^2) )]]];
Print["The energy levels of the ground and excited states are
", EigenSystem[[1]][[1]], "
and
", EigenSystem[[1]][[2]], "respectively."];
```

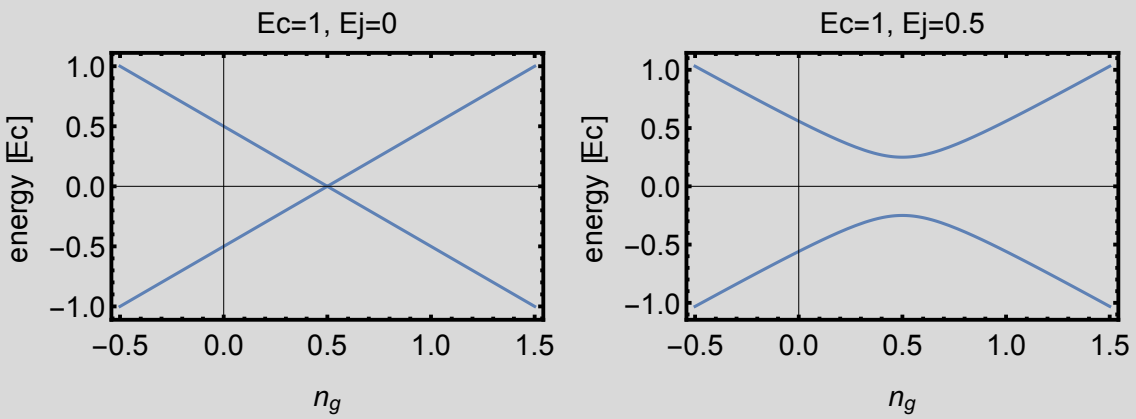
The energy levels of the ground and excited states are

$$-\frac{1}{2}\sqrt{Ej^2 + Ec^2\,(1 - 2\,ng)^2}$$

and

$$\frac{1}{2}\sqrt{Ej^2 + Ec^2\,(1 - 2\,ng)^2}$$ respectively.

```
Grid[{{
   Plot[{EigenSystem[[1]][[1]], EigenSystem[[1]][[2]]} /. {Ec → 1, Ej → 0}, {ng,
     -0.5, 1.5}, FrameLabel → {{"energy [Ec]", None}, {"n_g", "Ec=1, Ej=0"}}],
   Plot[{EigenSystem[[1]][[1]], EigenSystem[[1]][[2]]} /. {Ec → 1, Ej → 0.5},
    {ng, -0.5, 1.5},
    FrameLabel → {{"energy [Ec]", None}, {"n_g", "Ec=1, Ej=0.5"}}]
  }}]
```



(c) Plot the energy levels of H_{CPB} for $E_j = E_c = 2$ as a function of ng. Truncate your matrix after the lowest 10 charge states.

(i) we create the Hamiltonian for the 10 lowest states as requested and for the 60 lowest states, see (ii).

```
H = DiagonalMatrix[Table[Ec (n - ng)^2, {n, 0, 10}]] +
    DiagonalMatrix[Table[-Ej / 2, {n, 1, 10}], 1] +
    DiagonalMatrix[Table[-Ej / 2, {n, 0, 9}], -1];
H2 = DiagonalMatrix[Table[Ec (n - ng)^2, {n, -20, 40}]] +
    DiagonalMatrix[Table[-Ej / 2, {n, 1, 60}], 1] +
    DiagonalMatrix[Table[-Ej / 2, {n, 0, 59}], -1];
MatrixForm[
 H]
```

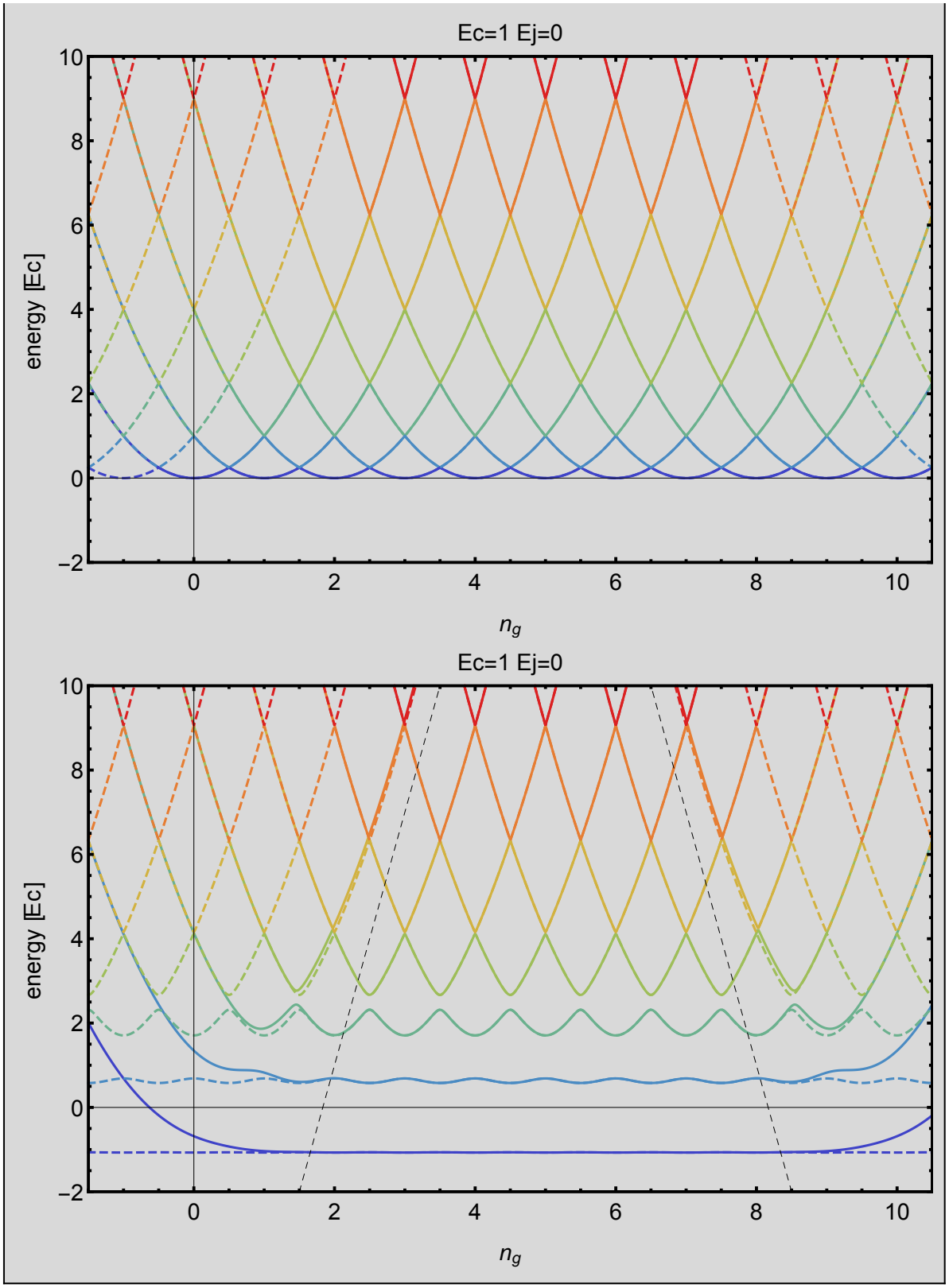
$$\begin{pmatrix}
\text{Ec ng}^2 & -\frac{\text{Ej}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{\text{Ej}}{2} & \text{Ec (1 - ng)}^2 & -\frac{\text{Ej}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{\text{Ej}}{2} & \text{Ec (2 - ng)}^2 & -\frac{\text{Ej}}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{\text{Ej}}{2} & \text{Ec (3 - ng)}^2 & -\frac{\text{Ej}}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{\text{Ej}}{2} & \text{Ec (4 - ng)}^2 & -\frac{\text{Ej}}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{\text{Ej}}{2} & \text{Ec (5 - ng)}^2 & -\frac{\text{Ej}}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{\text{Ej}}{2} & \text{Ec (6 - ng)}^2 & -\frac{\text{Ej}}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{\text{Ej}}{2} & \text{Ec (7 - ng)}^2 & -\frac{\text{Ej}}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\text{Ej}}{2} & \text{Ec (8 - ng)}^2 & -\frac{\text{Ej}}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\text{Ej}}{2} & \text{Ec (9 - ng)}^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\text{Ej}}{2}
\end{pmatrix}$$

(ii) We diagonalize and plot the eigenenergies as solid lines with colours corresponding to the lowest state (blue) to the highest state (red).

Because the real Hilbert space is infinite (in the number of charge states n) and we do truncate our hamiltonian after 10 states (H), its diagonalization is physically correct only in a certain energy range. This range is defined by the energy of the eigenstates converging to a stable value if n is increased. Therefore we also plot the eigenstates of the system with 60 charge states (H2) as dashed lines. The energy range in which diagonalization of H is valid is given by the points where the dashed and the solid lines overlap (i.e. within the black dashed lines.).

```
Ej = 0; Ec = 1;
EigenValues1 = Table[
    Sort[{ng, #} & /@ Eigenvalues[H], #1[[2]] < #2[[2]] &], {ng, -1.5, 10.5, 0.01}]^T;
EigenValues12 = Table[Sort[{ng, #} & /@ Eigenvalues[H2], #1[[2]] < #2[[2]] &],
    {ng, -1.5, 10.5, 0.01}]^T;
Ej = 2; Ec = 1;
EigenValues2 = Table[
    Sort[{ng, #} & /@ Eigenvalues[H], #1[[2]] < #2[[2]] &], {ng, -1.5, 10.5, 0.01}]^T;
EigenValues22 = Table[Sort[{ng, #} & /@ Eigenvalues[H2], #1[[2]] < #2[[2]] &],
    {ng, -1.5, 10.5, 0.01}]^T;
Clear[
 Ej,
 Ec]
```

```
Grid[{{
   Show[
    {Table[ListPlot[EigenValues1[[i]], PlotStyle → ColorData["Rainbow"][i / 7],
       Joined → True, PlotRange → {{-1.5, 10.5}, {-2, 10}}], {i, 1, 11}],
     Table[ListPlot[EigenValues12[[i]], PlotStyle →
        Directive[{ColorData["Rainbow"][i / 7], Dashed}], Joined → True,
       PlotRange → {{-1.5, 10.5}, {-2, 10}}], {i, 1, 30}]},
    AspectRatio → 0.6, ImageSize → 600, FrameLabel →
     {{"energy [Ec]", None}, {"n_g", "Ec=1 Ej=0"}}]
  }, {
   Show[
    {Table[ListPlot[EigenValues2[[i]], PlotStyle → ColorData["Rainbow"][i / 7],
       Joined → True, PlotRange → {{-1.5, 10.5}, {-2, 10}}], {i, 1, 11}],
     Table[ListPlot[EigenValues22[[i]], PlotStyle →
        Directive[{ColorData["Rainbow"][i / 7], Dashed}],
       Joined → True, PlotRange → {{-1.5, 10.5}, {-2, 10}}], {i, 1, 30}],
     Graphics[{{Dashed, Line[{{1.5, -2}, {3.5, 10}}]},
       {Dashed, Line[{{8.5, -2}, {6.5, 10}}]}}]},
    AspectRatio → 0.6, ImageSize → 600, FrameLabel →
     {{"energy [Ec]", None}, {"n_g", "Ec=1 Ej=0"}}]
  }}]
```


Ec=1 Ej=0
energy [Ec]
10
8
6
4
2
0
−2
0
2
4
6
8
10
n_g
Ec=1 Ej=0
energy [Ec]
10
8
6
4
2
0
−2
0
2
4
6
8
10
n_g

(d) For $E_J = E_C = 2$ and ng = 5.5, what are the transitions frequencies between the lowest 3 energy bands.

(i) from the diagonalization we obtain for the lowest 3 energylevels at $n_g = 5.5$:

```
levels = EigenValues2[[#]][601][[2]] & /@ {1, 2, 3};
"Energy levels are: " <> ToString[
  NumberForm[{levels[[1]], levels[[2]], levels[[3]]}, {3, 2}]] <> " in units of Ec"
```

```
Energy levels are: {-1.06, 0.58, 2.32} in units of Ec
```

(ii) therefore the lowest 3 transition frequencies for $n_g = 5.5$ are :

```
"Transition frequencies are: " <> ToString[NumberForm[{levels[[2]] - levels[[1]],
    levels[[3]] - levels[[1]],
    levels[[3]] - levels[[2]]} , {3, 2}] ] <> " in units of Ec"
```

```
Transition frequencies are: {1.64, 3.38, 1.74} in units of Ec
```

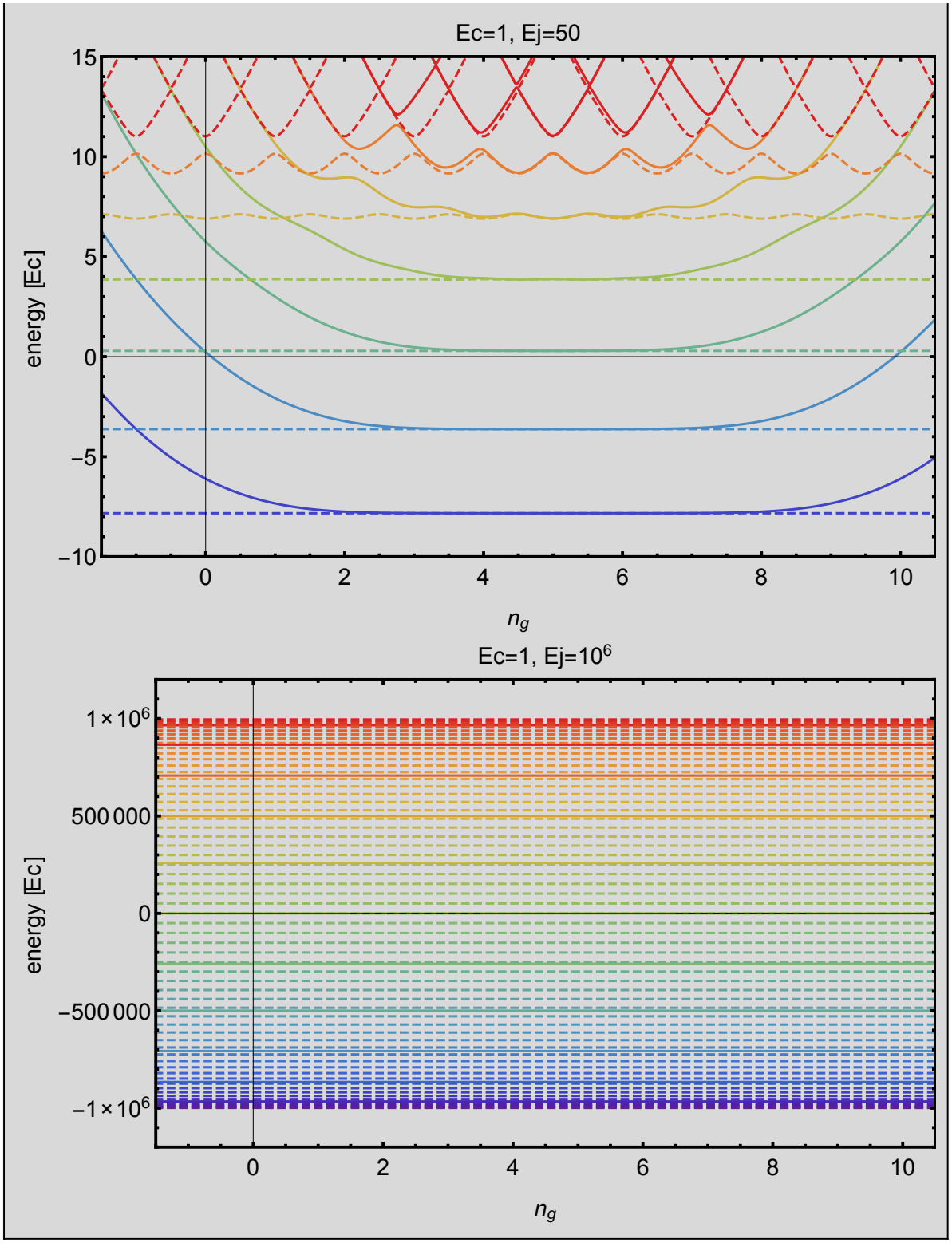
(e) Describe what happens to the dependence of the energy levels on ng for Ej→∞. In this limit, what are now the transitions frequencies between the lowest 3 energy bands.What is the energy difference between the lowest and highest energy band.

(i) For $E_J/E_c \to \infty$ the eigenenergies of the system (a combination of charging (Ec) and Josephsen (Ej) energies) get more and more independent of ng, because the off-diagonal elements in H (see above) dominate the diagonal terms. Therefore we expect for large Ej's 10 straight lines as a function of n_g. However, in the limit of Ej=∞ or Ec=0, the charge states are not good quantum numbers anymore, independent of the gate charge ng and we would need to take into account an infinite number of charge states to accurately diagonalize the system. For a finite number of states diagonalization is only valid for the states of low energies.

(ii) We plot eigenenergies for Ej/Ec=50 (close to the typical transmon limit) and $Ej/Ec = 10^6$ for both Hamiltonians H as solid lines (and H2 as dashed lines resp., to compare validity of numerical diagonalization)

```
Ej = 10; Ec = 1;
EigenValues3 = Table[
    Sort[{ng, #} & /@ Eigenvalues[H], #1[[2]] < #2[[2]] &], {ng, -1.5, 10.5, 0.01}]^T;
EigenValues32 = Table[Sort[{ng, #} & /@ Eigenvalues[H2], #1[[2]] < #2[[2]] &],
    {ng, -1.5, 10.5, 0.01}]^T;
Ej = 10^6; Ec = 1;
EigenValues4 = Table[
    Sort[{ng, #} & /@ Eigenvalues[H], #1[[2]] < #2[[2]] &], {ng, -1.5, 10.5, 0.01}]^T;
EigenValues42 = Table[Sort[{ng, #} & /@ Eigenvalues[H2], #1[[2]] < #2[[2]] &],
    {ng, -1.5, 10.5, 0.01}]^T;
Clear[Ej, Ec]


Grid[{{
   Show[
    {Table[ListPlot[EigenValues3[[i]], PlotStyle → ColorData["Rainbow"][i / 7],
       Joined → True, PlotRange → {{-1.5, 10.5}, {-10, 15}}], {i, 1, 11}],
     Table[ListPlot[EigenValues32[[i]], PlotStyle →
        Directive[{ColorData["Rainbow"][i / 7], Dashed}], Joined → True,
       PlotRange → {{-1.5, 10.5}, {-10, 15}}], {i, 1, 60}]},
    AspectRatio → 0.6, ImageSize → 600, FrameLabel →
     {{"energy [Ec]", None}, {"n_g", "Ec=1, Ej=50"}}]
  },
  {
   Show[{Table[ListPlot[EigenValues4[[i]],
       PlotStyle → ColorData["Rainbow"][i/10], Joined → True,
       PlotRange → {{-1.5, 10.5}, {-1.2 10^6, 1.2 × 10^6}}], {i, 1, 11}],
     Table[ListPlot[EigenValues42[[i]], PlotStyle →
        Directive[{ColorData["Rainbow"][i/60], Dashed}], Joined → True,
       PlotRange → {{-1.5, 10.5}, {-1.2 10^6, 1.2 × 10^6}}], {i, 1, 60}],
     Graphics[{{Dashed, Line[{{1.5, -2}, {3.5, 10}}]},
       {Dashed, Line[{{8.5, -2}, {6.5, 10}}]}}]},
    AspectRatio → 0.6, ImageSize → 600, FrameLabel →
     {{"energy [Ec]", None}, {"n_g", "Ec=1, Ej=10^6"}}]
  }}]
```



We observe that for increasing Ej a decreasing dependence on ng for increasing levels (from blue to red color), see upper plot. For a finite charging energy the total energy of the system increases to ∞ for increasing energy level n and the particle is bound (discrete energy levels).
However for Ej/Ec="∞", the total energy of the system is limited by +/- Ej. All (an infinite number) possible states are distributed between these two limits with increasing density of states toward +/- Ej. Therefore no discretization is possible and the particle is not bound anymore. As an analogy for the case Ec=0(Ej="∞") one can think of an harmonic oscillator potential with no kinetic energy term

(no momentum). Because of Heisenbergs uncertainty relation, the state is equally distributed over all space, i.e. not bound.

(iii) The correct answer is, that the lowest three levels are not defined for a system of infinite number of states (not bound), and therefore also not the transition frequencies. For comparison, we provide the energylevels for the reduced system of only 10 possible states and its transitions as a function of Ej.

```
"Energylevels:" <> ToString[NumberForm[
   {EigenValues4[[1]][[1]][[2]], EigenValues4[[2]][[1]][[2]], EigenValues4[[3]][[1]][[2]]} / 10^6,
   {3, 2}]] <> " in units of Ej"
"Transition frequencies:" <> ToString[
  NumberForm[{EigenValues4[[2]][[1]][[2]] - EigenValues4[[1]][[1]][[2]],
     EigenValues4[[3]][[1]][[2]] - EigenValues4[[2]][[1]][[2]],
     EigenValues4[[3]][[1]][[2]] - EigenValues4[[1]][[1]][[2]]} / 10^6,
   {3, 2}]] <> " in units of maximal Ej"
```

```
Energylevels:{-0.97, -0.87, -0.71} in units of Ej
```

```
Transition frequencies:{0.10, 0.16, 0.26} in units of maximal Ej
```

Exercise 2

(a) If we can only measure transmission and reflection of the resonator, how can we infer the state of the qubit? Does the amplitude of the drive (or equivalently the number of photons in the resonator) lead to problems? If yes, how to we need to choose it?

(i) H can be rewritten in terms of an effective resonance frequency of the resonator, i.e. $H = \hbar\omega_r^{\text{eff}}\left(a^\dagger a + 1/2\right) - \frac{\hbar\omega_q}{2}\sigma_z$, which describes the energy of a qubit and a resonator with frequency $\omega_r^{\text{eff}}=(\omega_r + \frac{g^2}{\Delta}\sigma_z)$. We can measure the state of the qubit by determining the resonance frequency of the resonator in a transmission measurement. The number of photons in the resonator does not matter as long as the system is well within the dispersive regime ($\omega_r - \omega_q >> g$). For too high photon numbers, the atom is driven off-resonantly and the non-demolition nature of the measurement is violated. For typical experiments the mean photon number of the drive can be as high as $< a^\dagger a > \sim 10$, depending on the parameters of the Jaynes-Cummings Hamiltonian.

(b) How does the qubit energy depend on the resonator state?

(i) The dispersive Hamiltonian can be rewritten as $H= \hbar\omega_r\left(a^\dagger a + \frac{1}{2}\right) + \sigma_z\left(-\frac{\hbar\omega_q}{2} + \frac{\hbar g^2}{\Delta}\left(a^\dagger a + \frac{1}{2}\right)\right) = \hbar\omega_r\left(a^\dagger a + \frac{1}{2}\right) + \sigma_z\frac{\hbar\omega_q^{\text{eff}}}{2}$ with the qubit energy $\hbar\omega_q^{\text{eff}} = -\hbar\omega_q + \frac{2\hbar g^2}{\Delta}\left(a^\dagger a + 1\right)$ depending linearly on the discrete number of photons (Fock states) $n=a^\dagger a$ in the resonator.

The qubit energy is a sum of

a) the bare qubit energy: $\hbar\omega_q$,

b) the Stark shift: $\frac{2g^2}{\Delta}\left(a^\dagger a\right)$ from the electric field in the resonator,

c) the Lamb shift: $\frac{g^2}{\Delta}(1)$ from the vacuum field in the resonator.

(c) Modulating the gate charge ng, see Exercise 1, with a pulse at a frequency ω_g we can change the state of the qubit. Using the transmission through the resonator, how can we infer the state of the resonator using the qubit? Again, does the amplitude of the drive lead to problems? If yes, how do we need to choose it?

(i) from (b) we observe, that the resonance frequency of the atom depends on the photon state of the resonator. A spectroscopic measurement, in which the frequency of the gate pulse (ng) is varied will therefore lead to a resonance peak at each frequency $-\omega_q + \frac{2g^2}{\Delta}(n+1)$, n being the number of photons in the resonator, with a strength that is proportional to population of the the n-photon Fock state of the resonator.

(ii)

(A)

A typical measurement with a single resonator in the strong dispersive regime (decay qubit $\gamma << \frac{g^2}{\Delta}$) and one qubit coupled to it could work as follows:

1) Prepare the photon state in the empty cavity, e.g. a thermal state or a superposition of several Fock states, and your qubit in the ground state.
2) Apply the gate pulse with fixed length and amplitude and varying frequency. If the gate pulse corresponds to a π-pulse (the frequency corresponds to ω_q^{eff}), the qubit is brought to the excited state.
3) Wait until the photon state decayed from the cavity, which is now empty ($\gamma << \kappa$!).
4) Send a probe tone through the cavity at resonance frequency $\omega_r^{\text{eff}}=\omega_r-\frac{g^2}{\Delta}$(see (a)). As a result, high transmission is measured if the qubit is in the ground state, and low transmission is measured if it is in the excited state, (i.e. the gate pulse corresponds to a π-pulse).
5) For this measurement the amplitude of the drive does not matter, as long as the approximations for the dispersive regime are met, see (a).

(B)

Recently groups change to systems with a qubit dispersively coupled to 2 resonators simultaneously. The state of one resonator can then be directly measured with a pulsed measurement on the other resonator, similar to (A), and it is not necessary to wait for the cavity state to decay.