

Homework Assignment for Circuit QED

If you have any questions or find typos, please contact Leo DiCarlo (l.dicarlo@tudelft.nl)

Problem 1: Dispersive circuit QED hamiltonian.

Consider a qubit coupled to a single-mode resonator. As Andreas described, the system is governed by the Jaynes-Cummings hamiltonian

$$H_{\text{JC}} = \hbar\omega_r \left(n + \frac{1}{2} \right) - \hbar\frac{\omega_a}{2}\sigma_z + \hbar g \left(a\sigma_+ + a^\dagger\sigma_- \right),$$

where a^\dagger and a are creation and annihilation operators for the mode, $n = a^\dagger a$, and σ_z , $\sigma_+ = \frac{1}{2}(\sigma_x - i\sigma_y)$ and $\sigma_- = \frac{1}{2}(\sigma_x + i\sigma_y)$ are qubit operators. To ensure we use the same conventions, let us specify the Pauli operators:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In this problem, we derive an approximate hamiltonian valid in the dispersive regime, where the qubit and resonator are off-resonant and their detuning $\Delta = \omega_a - \omega_r$ is large (in an absolute sense) relative to the dipolar coupling strength, i.e., $g/|\Delta| \ll 1$. Consider the unitary transformation:

$$U = \exp \frac{g}{\Delta} \left(a\sigma_+ - a^\dagger\sigma_- \right) = I + \frac{g}{\Delta} \left(a\sigma_+ - a^\dagger\sigma_- \right) + \frac{1}{2} \frac{g^2}{\Delta^2} \left(a\sigma_+ - a^\dagger\sigma_- \right)^2 + \dots$$

- As a warmup, show that the second order term of U can be rewritten as

$$-\frac{1}{2} \frac{g^2}{\Delta^2} \left(n - \frac{1}{2}\sigma_z + \frac{1}{2}I \right).$$

Hints: make use of the commutation relation $[a, a^\dagger] = 1$ and $\sigma_-^2 = \sigma_+^2 = 0$.

Consider the transformed hamiltonian given by

$$H = UH_{\text{JC}}U^\dagger.$$

We will approximate H by keeping only terms to first order in the small parameter g/Δ . Let us do this methodically by transforming each term in H_{JC} . The transformation of the vacuum energy is trivial:

$$U \left(\frac{1}{2}\hbar\omega_r \right) U^\dagger = \frac{1}{2}\hbar\omega_r$$

- Now on to the other resonator term:

$$U (\hbar\omega_r n) U^\dagger \approx \hbar\omega_r n + \hbar\omega_r \frac{g}{\Delta} \text{-----} + \hbar\omega_r \frac{g^2}{\Delta^2} \text{-----}$$

- Now on to the qubit term:

$$U \left(-\frac{1}{2} \hbar \omega_a \sigma_z \right) U^\dagger \approx -\frac{1}{2} \hbar \omega_a \sigma_z + \frac{1}{2} \hbar \omega_a \frac{g}{\Delta} \text{-----} + \frac{1}{2} \hbar \omega_a \frac{g^2}{\Delta^2} \text{-----}$$

- And finally to the interaction term:

$$U \hbar g \left(a \sigma_+ + a^\dagger \sigma_- \right) U^\dagger \approx \hbar g \left(a \sigma_+ + a^\dagger \sigma_- \right) + \hbar g \frac{g}{\Delta} \text{-----}$$

- Show that the combination of all these terms simplifies to the hamiltonian given in Lecture 1:

$$H \approx \hbar \omega_r \left(n + \frac{1}{2} \right) - \frac{1}{2} \hbar \left(\omega_a + \frac{g^2}{\Delta} \right) \sigma_z - \hbar \frac{g^2}{\Delta} n \sigma_z + \frac{1}{2} \hbar \frac{g^2}{\Delta}$$

- Identify the Lamb and AC-Stark shifts.
- Which term do we exploit to perform qubit readout in dispersive circuit QED?

Problem 2: *Resonator-mediated qubit-qubit interaction.*

Let us consider now a system of two qubits (labelled 1 and 2) that both couple to a common single-mode resonator. The qubits have no direct interaction between them. The Jaynes-Cummings hamiltonian is expanded into a Tavis-Cummings hamiltonian:

$$H_{\text{TC}} = \hbar \omega_r \left(n + \frac{1}{2} \right) - \sum_{i \in \{1,2\}} \hbar \frac{\omega_{a,i}}{2} \sigma_{z,i} + \hbar g_i \left(a \sigma_{+,i} + a^\dagger \sigma_{-,i} \right),$$

We assume both qubits are dispersive with respect to the resonator, i.e., $g_{1(2)}/|\Delta_{1(2)}| \ll 1$, where $\Delta_i = \omega_{a,i} - \omega_r$. As in the previous problem, we can derive an approximate hamiltonian for this regime, now using the transformation

$$U = \exp \left(\frac{g_1}{\Delta} \left(a \sigma_{+,1} - a^\dagger \sigma_{-,1} \right) + \frac{g_2}{\Delta} \left(a \sigma_{+,2} - a^\dagger \sigma_{-,2} \right) \right).$$

For this problem, let us focus on the transformation of the interaction term only:

$$U \left(\hbar g_1 \left(a \sigma_{+,1} + a^\dagger \sigma_{-,1} \right) + \hbar g_2 \left(a \sigma_{+,2} + a^\dagger \sigma_{-,2} \right) \right) U^\dagger$$

- Show that in addition to the terms for each qubit (similar to those in Problem 1), there appears a qubit-qubit interaction of the form

$$\hbar J (\sigma_{-,1} \sigma_{+,2} + \sigma_{+,1} \sigma_{-,2}).$$

Jargon: we say that the cavity *mediates* a *swap* (also called *exchange*) interaction between the two qubits.

- Give an expression for J in terms of g_1 , g_2 , Δ_1 , and Δ_2 .
Note: this expression for J will differ from that given in the lecture slides. The reason is that the transformations of the resonator and qubit terms combine to give more terms of this form.