Solutions to problem set for IDEA League QI School
Control and coherence of semiconductor spin qubits
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<u>Exercise 1:</u> Quantum dot energy scales (8 Points)

1. The eigenenergies of the z-dynamics are given by a the particle-in-a-box solutions:

$$E_n = \frac{\hbar^2 \pi^2}{2m^*} \frac{n^2}{d_z^2}.$$

   The energy difference between first and second state is thus:

$$E_2 - E_1 = \frac{\hbar^2 \pi^2}{2m^* d_z^2} (4 - 1) = 160 \text{ meV} = k_B \cdot 1900 \text{ K}$$

   with $m^* = 0.07 m_e$.

2. The energy spacing of a harmonic oscillator is given by $\hbar\omega = \hbar\sqrt{k/m^*}$. With $k = 2 \cdot 2 \text{ meV}/(20 \text{ nm})^2$, one obtains $\hbar\omega = 3.3$ meV = 38 K. A reasonable measure of the total width Δx is twice the rms-width of the harmonic oscillator ground state, i.e., $\Delta x = 2\sqrt{\frac{\hbar}{2m^*\omega}} = 26$ nm. The ground state wave function is a Gaussian, the first excited state its deerivative. Note 1: The transverse level splitting in GaAs quantum dots used as spin qubits is typically somehwat smaller (about 1 meV) and the wave functions are correspondingly wider. Note 2: For simplicity we have only discussed one coordinate. For a 2D quadratic potential, the x and y degrees of freedom decouple (for an appropriate rotation of the coordinate system) and the respective energies add.

3. The electrostatic potential anywhere outside a homogeneously charge sphere with charge e at distance r from the center is

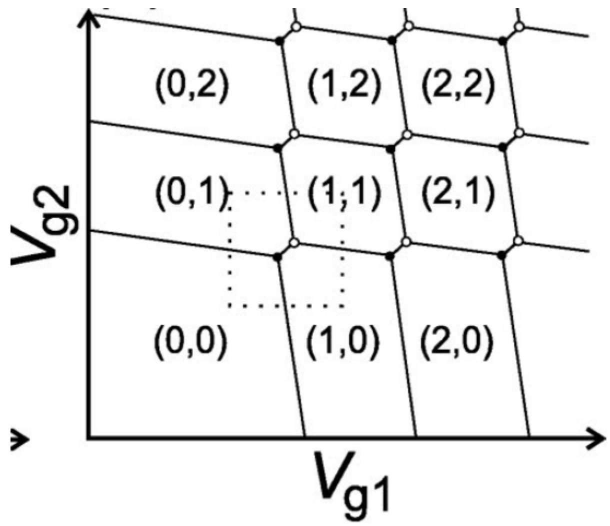
$$V(r) = \frac{e}{4\pi\epsilon_o\epsilon_r r}.$$

   Hence, bringing an electron from far away to the surface of the sphere costs an energy of 8.5 meV = 98 K (5.5 meV = 64 K) for $r = 13$ nm (20 nm) Note: In practice, typical charging energies of GaAs quantum dots are again about a factor of two smaller. One aspect we have neglected here is the presence of metallic gates, which cuts off the potential at a finite distance.

4. The interdot coupling energy is given by the Coulomb energy of two electrons 180 nm apart:

$$E_c = \frac{1}{4\pi\epsilon_0\epsilon_r} \frac{e^2}{d} = 610\ \mu\text{eV} = \text{k}_\text{B} \cdot 7.1\text{K}.$$

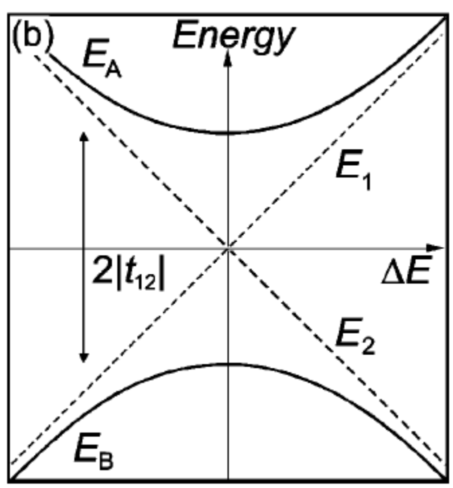
5. Diagram taken from Figure 2 of van er Wiel et al., Rev. Mod. Phys. **75**, p. 1 (2003).



The interdot coupling determines how much the transitions in one dot depend on the occupancy of the other, e.g., the distance between the (0, 1)-(0, 2) and the (1, 1)-(1, 2) transitions. Consequently, it also sets the length of the (1, 1)-(0, 2) boundary. (Numbers in parenthesis indicate the occupancy of the left and right dot.)

<u>Exercise 2:</u> Energy diagram of the charge qubit (8 Points)

1. Diagram taken from Figure 18 of van er Wiel et al., Rev. Mod. Phys. **75**, p. 1 (2003):



2.

$$H_0 = \frac{\epsilon}{2}\sigma_z + \frac{\Delta}{2}\sigma_x.$$

The first term corresponds to the detuning with energy difference ϵ between $|L\rangle$ and $|R\rangle$, the second term tunneling between these states. $\Delta = 2t_{12}$ in the diagram.

3. According to first order pertubation theory, the change in energy with a small variation $\delta\epsilon$ in detuning is $\delta E_i = \langle\psi_i|\delta H|\psi_i\rangle$ with H_0 as in the previous part and

$$\delta H = \frac{\delta\epsilon}{2}\sigma_z.$$

This leads to:

$$\begin{aligned} \delta E_i &= \frac{\delta\epsilon}{2}\langle\psi_i|\sigma_z|\psi_i\rangle \\ \Rightarrow \quad \frac{dE_i}{d\epsilon} &= \frac{\delta E_i}{\delta\epsilon} = \frac{1}{2}\langle\psi_i|\sigma_z|\psi_i\rangle. \end{aligned}$$

On the other hand we get

$$\begin{aligned} \langle\sigma_z\rangle = \langle\psi_i|\sigma_z|\psi_i\rangle &= \langle\psi_i|L\rangle\langle L| - |R\rangle\langle R|\psi_i\rangle \\ &= \langle\psi_i|L\rangle^2 + \underbrace{\langle\psi_i|R\rangle^2}_{=1-\langle\psi_i|L\rangle^2} \\ &= 2\langle\psi_i|L\rangle^2 - 1 \\ \Rightarrow \quad \langle\psi_i|L\rangle^2 &= \frac{\langle\sigma_z\rangle + 1}{2}. \end{aligned}$$

Thus we obtain the following expression for the probability of being in a $|L\rangle$ state:

$$\langle\psi_i|L\rangle^2 = \frac{dE_i}{d\epsilon} + \frac{1}{2}.$$

The slope of the energies thus reflects the probabilities of the eigenstates being on the left or right. For example:

$$\begin{array}{lllllll} \dfrac{dE_i}{d\epsilon} = 0 & \Rightarrow & \langle\psi_i|L\rangle^2 = \dfrac{1}{2} & \Rightarrow & \text{half } |L\rangle,|R\rangle \\ \dfrac{dE_i}{d\epsilon} > 0 & \Rightarrow & \langle\psi_i|L\rangle^2 > \dfrac{1}{2} & \Rightarrow & \text{more } |L\rangle \\ \dfrac{dE_i}{d\epsilon} < 0 & \Rightarrow & \langle\psi_i|L\rangle^2 < \dfrac{1}{2} & \Rightarrow & \text{more } |R\rangle \text{ in } |\psi_i\rangle. \end{array}$$

4. The variance of the capacitor voltage V_C in thermal equilibrium can be obtained from the equipatition theorem:

$$\langle V_c^2\rangle = \frac{k_B T}{C} = (1.2\ \mu\text{V})^2.$$

(C = 1 pF is the capactitance. Alternatively, one could also determine the spectrum of V_c arising from Nyquist-Johnson noise in the resistor and integrate over it.) The r.m.s.-variation of the qubit detuning is thus $\delta\epsilon = 0.12\ \mu$eV due to the capacitive lever arm of 1/10. The correlation time τ_c is just the RC time constant, giving $\tau_c = 1\ \mu$s.

Assuming that the voltage is quasistatic ($\tau_c \gg$ evolution times considered), we can use the relation $T_2^* = \frac{\hbar\sqrt{2}}{\delta\epsilon} = 7.8$ ns (c.f. slide 77 of the lectures). Since we find $T_2^* \ll \tau_c$, the quasistatic approximation is justified for all times of interest, and we are facing a T_2^*-effect. The correspondig coherence decay law is

$$\rho_{01}(t) = \rho_{01}(0)e^{-\left(\frac{t}{T_2^*}\right)^2}$$

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<u>Exercise 3:</u> Filter function of the free induction decay (6 Points)

As discussed in the lecture,

$$\langle\delta\phi^2(t)\rangle_\beta = \left\langle\left(\int_0^t dt\ \beta(t)\right)^2\right\rangle_\beta ,$$

where $\langle\cdot\rangle_\beta$ indicates averaging over the realizations of the noise process $\beta(t)$. Thus

$$\begin{aligned}
\langle\delta\phi^2(t)\rangle &= \int_0^t dt' \int_0^t dt''\ \langle\beta(t')\beta(t'')\rangle \\
&= \int \frac{d\omega}{\pi} S_\beta(\omega)\frac{F(\omega t)}{\omega^2}.
\end{aligned}$$

To obtain the last line, we have substituted

$$\langle\beta(t')\beta(t'')\rangle = \int \frac{d\omega}{2\pi} e^{-i\omega(t''-t')} S_\beta(\omega)$$

and defined the so called filter function

$$\begin{aligned}
F(\omega t) &= \frac{\omega^2}{2}\int_0^t dt' \int_0^t dt''\ e^{i\omega(t''-t')} \\
&= \frac{\omega^2}{2}\left|\int_0^t dt'\ e^{i\omega t'}\right|^2 \\
&= 2\sin^2\left(\frac{\omega t}{2}\right)
\end{aligned}$$